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NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* T.V. Narayana

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PERMANENT ADDRESS/*RÉSIDENCE FIXE*
20 Chatham Dr. N.W.
Calgary Alberta

NL-91 (3-74)

THE UNIVERSITY OF ALBERTA

A COMPARISON OF THE EFFECTIVENESS OF TOURNAMENTS

by

JAMES PATRICK HILL

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
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DEPARTMENT OF MATHEMATICS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled A COMPARISON OF THE EFFECTIVENESS OF TOURNAMENTS submitted by JAMES PATRICK HILL in partial fulfilment of the requirements for the degree of Master of Science.

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Supervisor

....

Earl Nordbrock

....

Date 3/9/74..........

ABSTRACT

The round robin tournament is introduced for which some basic theory is reviewed. A tournament of the "play the winner" type is introduced and compared numerically with the round robin and other common tournaments for the short duration case. For the moderate duration case, the "play the winner" tournament is modified and compared with the round robin via computer simulations and asymptotic behavior. The "play the winner" tournament is found to be far superior tc the round robin in terms of selecting the best player and selecting a subset containing the best player, and in addition exhibits a considerable saving in games. Some suggestions are given for improving the design , especially as regards its competence with unfavourable configurations of preference probabilities.

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The "play-the-winner" procedure developed in this thesis was selected on the basis of the results of an extensive computer simulation study performed in *APL* over the past six years. I thank Professor T. V. Narayana for generously making available the results of this study; I am also aware of my debt to all those involved in the development of both theory and simulations, notably Professor J. Zidek of U.B.C. and Professor K. W. Smillie.

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CHAPTER I

THE ROUND ROBIN TOURNAMENT

In the method of paired comparisons, t "objects" T_i $(i=1,2,\ldots,t)$ are presented in all pairwise comparisons from which one member from each pair is said to be "preferred". It can be seen that there are $\binom{t}{2}$ possible pairings. In general, the experiment may be repeated, for a total of say n replications. The results of the experiment may be expressed by means of a "score" for each object where the score of any object is the total number of times that object has been preferred in individual comparisons. It is then possible to define the "best" object as the one with the highest score.

It is convenient to consider such a paired comparison experiment in terms of a "Round Robin Tournament". In this context, the "objects" are known as "players", a player is "preferred" in a "comparison" if he has "won" a "game", and a player's score is obtained by awarding a point for each win.

The theory of round robin tournaments has been studied quite extensively by David [4]. In this chapter some of the results on the round robin are reviewed in order to provide a basis for comparing other types of tournament to the round robin. In succeeding chapters a different type of tournament developed by Narayana and Zidek [8] is described, and is compared to the round robin.

In order to insure that the concept of a "best" player is realistic, it is convenient to assume that the probabilities of the players beating one another satisfy a "linear" model. Suppose that each player T_i has true "strength" or "merit" V_i . The observed strength of T_i will vary from observation to observation and can be denoted by y_i , a continuous random variable where $-\infty \leq y_i \leq \infty$. In any one game, T_i wins over T_j if $y_i > y_j$. If it is possible to construct a scale of strength such that the probability of preference

$$\pi_{ij}=Pr(y_i-y_j>0)$$

can be expressed as $H(V_i-V_j)$ for all i,j , where $H(x)$ increases monotonically on $(-\infty,\infty)$ from 0 to 1 and where $H(-x)=1-H(x)$, then the π_{ij} are said to satisfy a "linear" model.

Consider a round robin tournament of n replications of all possible games between the players T_i $(i=1,2,\ldots,t)$. Define $x_{ij\gamma}$ as follows:

$$x_{ij\gamma}=\begin{cases}1 & \text{if } T_i \text{ beats } T_j\\ 0 & \text{if } T_j \text{ beats } T_i\end{cases} \quad (i,j=1,2,\ldots,t\ ;\ i\neq j\ ;\ \gamma=1,2,\ldots,n)$$

Ties are not permitted, there is no replication effect, and all games are independent. The equations

$$\Pr(x_{ij\gamma}=1)=\pi_{ij}$$

$$\Pr(x_{ij\gamma}=0)=\pi_{ji}=1-\pi_{ij}$$

define the preference probabilities π_{ij} . The score a_i of player T_i is given by

$$a_i = \sum_{\gamma=1}^{n} \sum_{j\neq i} x_{ij\gamma} = \sum_{\gamma=1}^{n} a_{i\gamma}$$

where $a_{i\gamma}$ denotes the score of T_i in the γ th replication.

Let α_{rs} $(r>s)$ be the number of times T_r beats T_s in n games. Given $C(\pi_{ij})$ the configuration of preference probabilities π_{ij} , Trawinski and David [13] arrived at an expression for the probability function of the vector of scores, namely

$$(1.1) \quad f(a; C(\pi_{ij})) = \sum \prod_{r>s}^{t} \binom{n}{\alpha_{rs}} \pi_{rs}^{\alpha_{rs}} \pi_{sr}^{n-\alpha_{rs}}$$

where the sum is taken over all α subject to the restrictions inherent in a . If all players are equal, then

$$f(a; C(\tfrac{1}{2})) = 2^{-\frac{1}{2}nt(t-1)} \sum \prod_{r>s}^{t} \binom{n}{\alpha_{rs}}$$

and

$$(1.2) \quad g(a; n) = \sum \prod_{r>s}^{t} \binom{n}{\alpha_{rs}}$$

gives the number of ways a can be realized. From this is derived the "partition" function $G(a; n)$ giving the number of permissible partitions of $\frac{1}{2}nt(t-1)$ into t

scores $a_1, \ldots, a_t$ irrespective of order. It can be seen that

$$G(a;\ n)=(t!/\prod_k m_k!)g(a;\ n) .$$

where m_k is the number of scores all of magnitude a_k. Tables of $G(a;\ n)$ are given in David [4].

In addition, for $d=(d_1, \ldots, d_{t-1})$ where $d_i=a_i-a_t$ $(i=1,2,\ldots,t-1)$, d_i has been shown to have mean

$$(1.3) \qquad \delta_i=n(\sum_{k=2}^{t} \pi_{ik} - \sum_{k=1}^{t-1} \pi_{tk})$$

and variance

$$(1.4) \qquad \sigma_{d_i d_i} = n[\sum_{k=2}^{t-1} (\pi_{ik}\pi_{ki}+\pi_{tk}\pi_{kt})+4\pi_{it}\pi_{ti}]$$

with covariances

$$(1.5) \qquad \sigma_{d_i d_j} = n[\sum_{k=1}^{t-1} \pi_{tk}\pi_{kt}+(\pi_{it}\pi_{ti}+\pi_{jt}\pi_{tj}-\pi_{ij}\pi_{ji})]$$

There are two major methods of approach concerning the selection of the best player which deserve review. The first involves calculating the smallest number n of replications required to ensure that with a given probability, the strongest player will obtain the highest score. The other is concerned with selecting the smallest subset of players which contains the strongest player with a given probability. Both types of approach were developed in the present context by Trawinski and David [13].

It is desirable to keep in mind the concept of a "least favourable configuration" of the preference probabilities $C(\pi_{ij})$. A configuration C may be considered as least favourable if the probability of correct selection attains a minimum with the configuration C. This may be thought of as being analogous to the "minimax" principle of game theory.

In particular, the case of the "single strong outlier" deserves mention. In this case, the outlier beats his $t-1$ equal opponents with probability $p>\frac{1}{2}$. David [4] has shown that this configuration is not least favourable for the selection of the strongest player, whereas with $p=\frac{1}{2}$ it is least favourable for subset selection. However, since least favourable configurations are difficult to find for realistic situations, and as the outlier case is of considerable importance as a "practical" least favourable case, this configuration will frequently be employed. The results obtained must therefore be considered in this light.

It is first desired to find the minimum number n of replications which will ensure that the best player obtains the highest score with probability P. In addition to the linear model, the single strong outlier configuration is assumed. This can be written

$$(1.6)\qquad \begin{aligned} &\pi_{tj} = \pi > \tfrac{1}{2}, && j=1,2,\ldots,t-1 \\ &\pi_{ij} = \tfrac{1}{2}, && i,j=1,2,\ldots,t-1\ ;\ i\neq j \end{aligned}$$

Under (1.6), equations (1.1) and (1.2) permit the joint distribution of scores to be written

$$f(a_{(i)}) = 2^{-n\binom{t-1}{2}} g(a;\ n)\pi^{a_{(t)}}(1-\pi)^{n(t-1)-a_{(t)}}$$

To account for ties, suppose m players share the highest score. There are $\frac{G(a;\ n)}{g(a;\ n)}$ permutations of the scores $a_1,\ldots,a_t$ of which a proportion m/t must have a top score associated with $T_{(t)}$. From (1.6) it can be seen that this contributes

$$(1.7)\qquad \frac{1}{m}\, 2^{-n\binom{t-1}{2}} \frac{m}{t}\, G(a;\ n)\pi^{a_{(t)}}(1-\pi)^{n(t-1)-a_{(t)}}$$

to P, the probability of correct selection. P itself is obtained by summing (1.7) over all $a_{(t)}$ which can be maximum and over all permissible values of the other scores. This gives

$$P = 2^{-n\binom{t-1}{2}} \sum_{a_{(t)}=c}^{n(t-1)} \pi^{a_{(t)}}(1-\pi)^{n(t-1)-a_{(t)}}$$

$$\cdot \sum \left(\frac{1}{t}\right) G(a_1,\ldots,a_t;\ n)$$

where the last sum extends over all a such that

$$\sum_{i=1}^{t-1} a_{(i)} = n\binom{t}{2} - a_{(t)}$$

and where c is the smallest integer greater than or equal to $\frac{1}{2}n(t-1)$.

In order to construct tables of n , Trawinski and David have developed an asymptotic approximation. The differences $d_{(i)}=a_{(i)}-a_{(t)}$, $(i=1,2,\ldots,t-1)$ are identically distributed equi-correlated variates. From (1.3) - (1.5) it is apparent that

$$\delta = -nt(\pi-\tfrac{1}{2})$$

$$\sigma_d^2 = n\,[(t+2)\pi(1-\pi)+\tfrac{1}{4}(t-2)] \tag{1.8}$$

$$\rho\,\sigma_d^2 = n\,[(t+1)\pi(1-\pi)-\tfrac{1}{4}]$$

As $n \to \infty$, the probability of ties tends to 0 , so the probability of correct selection is

$$\begin{aligned} P &= \lim_{n\to} \Pr\left\{d_{(i)}<0 \;;\; i=1,\ldots,t-1\right\} \\ &= \lim_{n\to} \Pr\left\{v_i < \Delta \;;\; i=1,\ldots,t-1\right\} \end{aligned}$$

where $v_i=(d_{(i)}-\delta)/\sigma_d$ and $\Delta=-\delta/\sigma_d$. The v_i have a limiting multivariate normal distribution, giving

$$P = (2\pi)^{-\frac{1}{2}(t-1)}\,|R|^{-\frac{1}{2}}\int_{-\infty}^{\Delta}\int_{-\infty}^{\Delta}\cdots\int_{-\infty}^{\Delta}\exp[-\tfrac{1}{2}v'R^{-1}v]\,dv \tag{1.9}$$

where R is the correlation matrix of the v_i , of the form

$$\begin{bmatrix} 1 & \rho \ldots & \rho \\ \rho & 1 & \ldots \\ & \ldots & \rho \\ \rho \ldots & \rho & 1 \end{bmatrix}$$

To simplify (1.9), bearing in mind that the v_i are equi-correlated, express

$$d_{(i)}=y_i-y_t\ ,\ i=1,2,\ldots,t-1$$

with all y's mutually independent. Take $E(y_i)=0$ and $E(y_t)=-\delta$. Note that

$$\sigma_d^2 = \text{var}\ y_i + \text{var}\ y_t$$

and

$$\rho\sigma_d^2 \equiv \text{cov}(d_i,\ d_j) = \text{var}\ y_t$$

Finally, taking $\text{var}\ y_i=(1-\rho)\sigma_d^2$, $\text{var}\ y_t=\rho\sigma_d^2$, the y's are completely specified. They may be standardized by setting

$$u_i= y_i/[(1-\rho)^{\frac{1}{2}}\sigma_d]$$

$$u_t= (y_t+\delta)/\rho^{\frac{1}{2}}\sigma_d$$

Thus

$$v_i= (y_i-y_t-\delta)/\sigma_d = (1-\rho)^{\frac{1}{2}}u_i-\rho^{\frac{1}{2}}u_t$$

Then

$$P = \lim_{n\to\infty} \Pr\left\{u_i<[\rho/(1-\rho)]^{\frac{1}{2}}u_t-\delta/[(1-\rho)^{\frac{1}{2}}\sigma_d],\ i=1,2,\ldots,t-1\right\}$$

$$=\int_{-\infty}^{\infty}[\Pr\{u_i<U_t|u_t\}]^{t-1}\varphi(u_t)du_t$$

where

$$U_t = [\rho/(1-\rho)]^{\frac{1}{2}} u_t - \delta/[(1-\rho)^{\frac{1}{2}} \sigma_d]$$

and

$$\varphi(u) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}u^2}$$

Thus

$$(1.10) \qquad P = \int_{-\infty}^{\infty} [\Phi(U_t)]^{t-1} \varphi(u_t) du_t$$

where Φ is the unit normal c.d.f.

It should be pointed out that this expression is valid only for $\rho > 0$, whereas the matrix R is positive definite for all $\rho > -1/(t-2)$. This is not a serious restriction as exact theory can easily be used in cases where $\rho < 0$.

The tables of n constructed by Trawinski and David [13] will be used in succeeding chapters to compare the round robin with other tournaments.

The second major proceedure of this type is that of selecting a subset of the players so as to include the strongest player with at least a prespecified probability P. The selection is made according to the following rule: Only those players T_i are retained in the subset for which $a_i \geq a_{max} - \nu$, where a_{max} is the highest score and ν, a non-negative integer, is chosen just large enough so that

$$P_{cs} = Pr\{a_{(t)} \geq a_{max} - \nu\} \geq P$$

Of course, this depends on the configuration of preference probabilities $C(\pi_{ij})$. However, it has been shown by Trawinski and David [13] that the configuration $C(\frac{1}{2})$ is least favourable in the sense that P_{cs} is minimized by $C(\frac{1}{2})$ over all $C(\pi_{ij})$ satisfying a linear model. Here, T_t is "tagged" as the strongest player.

Values of ν may be computed from the formula for $P_{cs}\ C(\frac{1}{2})$. This can be expressed as

$$
\begin{aligned}
P_{cs} &= \sum \Pr\left\{a_{(t)} \geq a_{max} - \nu \mid a,\ C(\tfrac{1}{2})\right\} p(a;\ C(\tfrac{1}{2})) \\
&= 2^{-\frac{1}{2}nt(t-1)} \sum \left[\Pr\left\{a_{(t)} \geq a_{max} - \nu \mid a,\ C(\tfrac{1}{2})\right\} G(a;\ n)\right]
\end{aligned}
$$

where $p(a;\ C(\frac{1}{2}))$ is the partition probability function, $G(a;\ n)$ is the partition function introduced earlier, and the sum runs over all distinct partitions a of $\frac{1}{2}nt(t-1)$ To evaluate the quantity summed, consider a_{i_t} as the score of $T_{(t)}$ in the i th possible permutation. Then the quantity summed is the frequency of permutations for which $a_{i_t} \geq a_{max} - \nu$. For each distinct value of a_{i_t}, the proportion of permutations is m_{i_t}/t where m_{i_t} is the number of scores tied with a_{i_t}. Thus the summed quantity is $M(a;\nu)G(a;\ n)/t$, where $M(a;\nu)$ is the number of scores in a which exceed or equal $a_{max} - \nu$. Given n, t, and ν,

$$
P_{cs}\left\{C(\tfrac{1}{2})\right\} = t^{-1}\, 2^{-\frac{1}{2}nt(t-1)} \sum M(a;\nu)\, G(a;\ n)
$$

The value of ν may be approximated asymptotically in a manner similar to that employed earlier. Introducing a continuity correction on ν , it is seen that

$$P_{cs} = \lim_{n\to\infty} \Pr\left\{a_{max} - a_{(t)} < \nu + \tfrac{1}{2}\right\}$$

$$= \lim_{n\to\infty} \Pr\left\{d_{(i)} < \nu + \tfrac{1}{2}\ ;\ i=1,2,\ldots,t-1\right\}$$

where $d_{(i)}$ is defined as before. Equations (1.8) take the form

$$\delta = 0$$

$$\sigma_d^2 = \tfrac{1}{2}nt$$

$$\rho\sigma_d^2 = \tfrac{1}{4}nt$$

and, setting $\rho=\frac{1}{2}$, it follows that

$$P_{cs} = \int_{-\infty}^{\infty} [\Phi(u_t+w)]^{t-1}\ \varphi(u_t)\,du_t$$

where

$$w = 2(\nu+\tfrac{1}{2})/(nt)^{\frac{1}{2}}$$

and the notation is the same as in (1.10) . Values of w as solutions of this expression have been tabulated by Bechhofer [1] and so asymptotic approximations to ν can easily be obtained.

Using the tables of Trawinski and David drawn up by these two methods, succeeding chapters will compare a new tournament type first suggested by Narayana and

Zidek [8] with the round robin both on the basis of selecting the best player and on the basis of selecting the smallest subset containing the best player.

CHAPTER II

COMPARISONS OF SHORT DURATION TOURNAMENTS

It is proposed in this chapter to introduce a "play-the-winner" tournament whose effectiveness will then be compared with that of the round robin and other tournaments in the case of a small maximum number of games. The case of tournaments of moderate duration will be considered in Chapter III .

From the conclusions of Narayana and Zidek [8,9], as well as the success of "play-the-winner" procedures in selecting the best of several binomial populations (Sobel and Weiss [11]), the basic procedure adopted here, called KOT , is as follows: at trial 1 , two of the n players are chosen at random. At trial i , $(1<i<n-1)$, the winner of trial $(i-1)$ meets one of the $(n-1)$ players who have received byes so far, so that at the end of $(n-1)$ trials or one knock out replication, every player has played at least once. At each stage, a player's score is deemed to be the difference between the number of games won and lost. The last winner of replication 1 now meets one of the $(n-1)$ other players at random to begin replication 2 , and in general, the last winner of replication $k-1$ meets one of the $(n-1)$ other players at random to begin replication k . At the termination of the tourna-

ment, the player with the highest score is said to be the winner of the tournament.

Assuming the case of a single strong outlier, it can be seen that the outlier's score tends to increase as the outlier plays a greater number of games. That is, the outlier's probability of winning the tournament varies with the proportion of games in which the outlier has participated. The round robin is a fully balanced tournament in the sense that every pair of players meets an equal number of times,and therefore the outlier plays no more than any other player. The design of KOT encourages the strong player to play more. This will be seen more explicitly in the asymptotic results of Chapter III .

It is desirable, however, to incorporate some balance in KOT with respect to the players other than the outlier, that is, every pair of equal players should meet approximately the same number of times. The obvious method by which to attempt to introduce this partial balance into KOT would be to state that after a replication, the last winner meets a player chosen at random from among those he has played least. However, in the cases of interest this is essentially equivalent to a round robin. The partially balanced variant of KOT in fact employed, which is called BKOT is defined as follows: at the end of replication i , the last winner has either

a) the highest score or b) has not the highest score. In case a) the last player continues to play with balance, that is, every player meets a randomly chosen player from among those he has played least, preference among those being given to the player who has played least in total. A random choice is made if no such preference exists in total plays. In case b) the last winner plays against one of the players with the highest score whom he has played the least, the same preferences (or lacking these the same random choice) as in a) above being used. Thus BKOT may be thought of as a balanced tournament among the equal players, combined with an equal number of meetings between the outlier and each of his opponents.

In order to increase still further the probability of the strongest player winning the tournament, elimination of poor players is introduced. By eliminating poor players it is expected to increase the proportion of plays concerning the outlier still further. To this end the following elimination rule is suggested for BKOT in the short duration case where the tournament consists of a maximum of m games.

A player, other than the winner of a game, is eliminated after the (m-k)th game if the difference d between the highest score and his own equals or exceeds f(k) , a pre-specified function. An eliminated player no longer plays in the tournament and a replication is supposed to be complete even though he has not played since the end of the previous replication.

The elimination function f(k) should be chosen so as to optimize the advantages of a fast elimination against the possible elimination of the strongest player. For small numbers of games $f(k)=2k+1$ and $f(k)=2k$ suggest themselves for the following reasons. Consider $f(k)=2k+1$. In any one game a player may decrease d by at most two by beating a lone top scorer. Thus, if a player is farther behind than twice the number of games remaining, he cannot possibly have top score at the end of m games. It would then be realistic to eliminate such a player (for whom a win is mathematically impossible) and concentrate in the remaining games on further comparisons between the possible winners. The reasoning for $f(k)=2k$ is similar, noting that $f(k)=2k+1$ is overly conservative. For example, the existence of a third player for whom $d\le\min(k-1,2k-3)$ would make it impossible for a player 2k behind the leader to overtake him. This can be seen if the two cases are inspected. If $d\le k-1$, then the best the low player can do is win the remaining k games against the top scorer. Then, however, the third player will have a higher score. If $d\le 2k-3$, then the low score can play at most two games against the top score before being required to play the third player. Thus he cannot play all games against the top scorer and so cannot catch up.

In the longer run, suppose there are k games remaining among n players in an integral number of replications. For n players the number of replications remaining is $k/n-1$. The maximum number of wins for any player is k . The maximum number of losses for any player is k/n-1 . From this, the following result is immediate.

Theorem 2.1

If k games remain in an integral number of replications of BKOT among n players, then the maximum deficit which can be made up is $\frac{k \cdot n}{n-1}$.

It should be noted that if the number of games remaining does not amount to an integral number of complete replications, the expression in Theorem 2.1 will be increased by at most two. The elimination rule suggested by Theorem 2.1 is clearly the best of those discussed. However, for computational reasons, the 2k and 2k+1 rules were employed for the calculations discribed below.

For simplicity, BKOT employing the elimination function f(k) will be denoted by B(f(k)) . After m games, the player with the highest score is declared the winner of the tournament. In this chapter, the most recent winner among two or more equal high scores is declared the winner of the tournament.

A numerical comparison of six common tournament types involving four players and as many as nine games has been performed by Glenn [6]. The effectiveness of these tournaments was rated both on the probability of the strongest player winning and on the number of games required. The tournaments included the following: R round robin tournament, K_0 single knock out tournament with a random draw and each pair playing one game, and K_1 single knock out with a random draw and each pair playing until one player has won two games. The probabilities of the strongest player winning and the expected duration of the tournament were computed for the range of examples in Table 1, which covers the case of the single strong outlier as well as more general cases.

In order to compare these tournaments with BKOT, the computer programme in Appendix I was devised to calculate the corresponding probabilities and expectations for the tournament BKOT using the elimination rules 2k and 2k+1 for four players and a number of maximum durations. The results are tabulated in Tables 2-7. It should be noted that K_o and K_1 are most efficient when the number of players is a power of two, so this comparison may be considered as a rather unfavourable comparison for BKOT against K_o and K_1.

It is seen that B(2k) is in general superior to B(2k+1). B(2k) is seen to compare reasonably well to K_1 with respect to choosing the best player, where both tournaments run a maximum of nine games. B(2k) where m=10 is clearly better than K_1 in this respect, and K_1 cannot easily be adapted to take advantage of an extra permissible game.

In order to further the comparison, the cost function

Table 1

Range of Examples

	π_{12}	π_{13}	π_{14}	π_{23}	π_{24}	π_{34}
0.1	.5500	.5500	.5500	.5000	.5000	.5000
0.2	.6000	.6000	.6000	"	"	"
0.3	.6500	.6500	.6500	"	"	"
0.4	.7000	.7000	.7000	"	"	"
0.5	.7500	.7500	.7500	"	"	"
1.1	.5400	.6500	.8600	.5000	.5000	.5000
1.2	"	"	"	.6400	.8500	.6200
1.3	"	"	"	.6127	.8396	.7679
1.4	"	"	"	.6400	.8500	.8200
1.5	"	"	"	.6500	.8600	.8400
2.1	.7000	.7600	.8600	.5000	.5000	.5000
2.2	"	"	"	.5758	.7247	.6598
2.3	"	"	"	.7500	.8200	.7200
2.4	"	"	"	.7500	.8400	.8000
2.5	"	"	"	.7500	.8500	.8400
3.1	.8000	.8500	.9000	.5000	.5000	.5000
3.2	"	"	"	.5862	.6923	.6136
3.3	"	"	"	.7000	.7500	.7000
3.4	"	"	"	.8000	.8300	.7500
3.5	"	"	"	.8400	.8700	.8500

Table 2

Probability of Strongest Player Winning R, K_o, and K_1

	R	K_o	K_1
0.1	.3197	.3025	.3303
0.2	.3968	.3600	.4199
0.3	.4795	.4225	.5159
0.4	.5657	.4900	.6147
0.5	.6531	.5625	.7119
1.1	.5239	.4581	.5374
1.2	.4615	.4249	.4789
1.3	.4564	.4209	.4737
1.4	.4503	.4173	.4688
1.5	.4472	.4156	.4667
2.1	.6910	.5958	.7406
2.2	.6637	.5817	.7179
2.3	.6438	.5734	.7063
2.4	.6393	.5707	.7034
2.5	.6370	.5694	.7022
3.1	.8190	.7216	.8751
3.2	.8028	.7133	.8652
3.3	.7924	.7089	.8602
3.4	.7803	.7044	.8562
3.5	.7734	.7014	.8540

Table 3

Probability of Strongest Player Winning B(2k)

	m				
	6	7	8	9	10
0.1	.3121	.3213	.3251	.3268	.3307
0.2	.3809	.4009	.4094	.4134	.4218
0.3	.4555	.4872	.5008	.5071	.5202
0.4	.5348	.5779	.5964	.6049	.6218
0.5	.6174	.6699	.6925	.7023	.7212
1.1	.4921	.5257	.5397	.5499	.5607
1.2	.4473	.4663	.4742	.4791	.4847
1.3	.4440	.4626	.4704	.4755	.4817
1.4	.4394	.4569	.4642	.4681	.4745
1.5	.4371	.4541	.4611	.4643	.4709
2.1	.6520	.7063	.7296	.7407	.7583
2.2	.6350	.6836	.7050	.7158	.7317
2.3	.6220	.6660	.6864	.6932	.7094
2.4	.6190	.6623	.6825	.6888	.7054
2.5	.6175	.6605	.6806	.6865	.7034
3.1	.7827	.8405	.8650	.8743	.8890
3.2	.7737	.8289	.8532	.8632	.8775
3.3	.7676	.8210	.8454	.8547	.8692
3.4	.7604	.8114	.8359	.8438	.8591
3.5	.7561	.8059	.8304	.8374	.8535

Table 4

Probability of Strongest Player Winning B(2k+1)

			m		
	6	7	8	9	10
0.1	.3103	.3198	.3212	.3249	.3290
0.2	.3769	.3984	.4013	.4095	.4183
0.3	.4491	.4846	.4889	.5017	.5153
0.4	.5266	.5763	.5817	.5984	.6159
0.5	.6064	.6705	.6764	.6957	.7153
1.1	.4866	.5236	.5257	.5425	.5538
1.2	.4436	.4644	.4683	.4758	.4823
1.3	.4400	.4617	.4652	.4721	.4797
1.4	.4351	.4563	.4598	.4651	.4732
1.5	.4326	.4536	.4570	.4616	.4700
2.1	.6412	.7080	.7132	.7338	.7520
2.2	.6242	.6864	.6933	.7102	.7275
2.3	.6088	.6701	.6792	.6906	.7083
2.4	.6055	.6669	.6762	.6864	.7050
2.5	.6037	.6653	.6746	.6842	.7034
3.1	.7704	.8471	.8515	.8698	.8849
3.2	.7610	.8366	.8427	.8593	.8744
3.3	.7537	.8295	.8376	.8518	.8673
3.4	.7443	.8210	.8318	.8428	.8590
3.5	.7388	.8162	.8285	.8374	.8547

Table 5

Expected Duration of R , K_1

	R	K_1
0.1	6.8694	7.4921
0.2	6.8503	7.4670
0.3	6.8148	7.4227
0.4	6.7607	7.3573
0.5	6.6863	7.2695
1.1	6.8175	7.3301
1.2	6.7320	7.2401
1.3	6.7740	7.2255
1.4	6.7757	7.1962
1.5	6.7769	7.1808
2.1	6.6588	7.2080
2.2	6.6406	7.1741
2.3	6.6197	7.0836
2.4	6.6298	7.0502
2.5	6.6355	7.0298
3.1	6.5120	7.0202
3.2	6.4683	6.9984
3.3	6.4611	6.9468
3.4	6.4523	6.8685
3.5	6.4547	6.7936

Table 6

Expected Duration of B(2k)

			m		
	6	7	8	9	10
0.1	5.5553	6.4295	7.2173	8.1656	9.1915
0.2	5.5340	6.4078	7.1848	8.1263	9.1373
0.3	5.4954	6.3666	7.1246	8.0545	9.0364
0.4	5.4370	6.3032	7.0321	7.9443	8.8835
0.5	5.3556	6.2141	6.9011	7.7846	8.6571
1.1	5.4718	6.3350	7.0846	8.0028	8.9661
1.2	5.3424	6.2307	6.9344	7.8580	8.7452
1.3	5.3359	6.2317	6.9364	7.8695	8.7470
1.4	5.3195	6.2185	6.9185	7.8539	8.7202
1.5	5.3111	6.2117	6.9094	7.8460	8.7063
2.1	5.3141	6.1668	6.8334	7.7069	8.5510
2.2	5.2722	6.1387	6.7961	7.6809	8.5069
2.3	5.2263	6.1005	6.7470	7.6403	8.4471
2.4	5.2175	6.0951	6.7404	7.6376	8.4394
2.5	5.2129	6.0923	6.7370	7.6363	8.4351
3.1	5.1159	5.9460	6.5098	7.3327	8.0381
3.2	5.0957	5.9351	6.4974	7.3301	8.0307
3.3	5.4954	6.3666	7.1246	8.0545	9.0364
3.4	5.4370	6.3032	7.0321	7.9443	8.8835
3.5	5.3556	6.2141	6.9011	7.7846	8.6571

Table 7

Expected Duration of B(2k+1)

			m		
	6	7	8	9	10
0.1	5.6183	6.7738	7.5521	8.5117	9.5043
0.2	5.5984	6.7541	7.5275	8.4742	9.4605
0.3	5.5626	6.7162	7.4814	8.4040	9.3796
0.4	5.5080	6.6556	7.4089	8.2954	9.2540
0.5	5.4318	6.5683	7.3041	8.1348	9.0599
1.1	5.5397	6.6872	7.4495	8.3550	9.3203
1.2	5.4217	6.5961	7.3627	8.2051	9.1607
1.3	5.4171	6.5967	7.3695	8.2091	9.1722
1.4	5.4024	6.5845	7.3597	8.1907	9.1533
1.5	5.3947	6.5781	7.3549	8.1811	9.1437
2.1	5.3927	6.5211	7.2499	8.0569	8.9714
2.2	5.3555	6.4981	7.2372	8.0267	8.9438
2.3	5.3151	6.4659	7.2167	7.9833	8.8967
2.4	5.3077	6.4610	7.2152	7.9778	8.8936
2.5	5.3038	6.4584	7.2149	7.9751	8.8918
3.1	5.2066	6.2913	6.9840	7.6710	8.5253
3.2	5.1896	6.2835	6.9865	7.6652	8.5214
3.3	5.1750	6.2739	6.9865	7.6569	8.5135
3.4	5.1551	6.2585	6.9845	7.6429	8.4999
3.5	5.1434	6.2492	6.9840	7.6357	8.4942

expected cost=q[1-Pr(C wins)]+expected number of games

may be introduced, where q is the assessed cost of a wrong decision, expressed in units of the cost per game, and C is the strongest player.

It is now possible to find for each example, two critical values of q,

q_1 : above which $B(2k)$ is preferable to K_0

q_2 : below which $B(2k)$ is preferable to K_1

Using the tournament $B(2k)$ with $m=7$, the critical values q_1 and q_2, rounded to the nearest integer below the exact vaue, are given in Table 8. It can be seen that for more than half the examples, there is a range of q in which $B(2K)$ is preferable to both K_0 and K_1, suggesting that BKOT is a reasonable alternative to these tournaments even when the number of players is a power of two. When the number of players is not a power of two, the knock out tournaments must be modified by introducing byes. This may be expected to decrease the efficiency of the knock out relative to that of BKOT. To test this, it is reasonable to compare BKOT to the knock out in the three player case.

Let the single outlier beat his equal opponents with probability p. Let $K(k,\ell)$ denote the knock out with a maximum of k games in the first round and ℓ in the second round. Here k and ℓ are of the form $2n-1$

Table 8

Critical values of q

	q_1	q_2		q_1	q_2		q_1	q_2
1.1	49	85	2.1	28	30	3.1	24	31
1.2	79	80	2.2	30	30	3.2	25	29
1.3	77	88	2.3	33	23	3.3	25	26
1.4	77	73	2.4	33	23	3.4	27	20
1.5	83	76	2.5	33	22	3.5	27	18

and a player advances to the second round by winning n games in the first. Denote by $Q_k(p)$ the probability of the outlier winning a round of k games. It is a straightforward calculation to see that

$$Q_3(p) = 3p^2 - 2p^3$$

$$Q_5(p) = 10p^3 - 15p^4 + 6p^5$$

$$Q_7(p) = 35p^4 - 84p^5 + 70p^6 - 20p^7$$

With a maximum of k games in the first round and ℓ in the second, the probability of the outlier winning the tournament is

$$\frac{2}{3}Q_k(p)Q_\ell(p) + \frac{1}{3}Q_\ell(p) \quad .$$

Following this method, the probabilities of the strong player winning in a maximum of 6, 8, 10, and 12 games have been calculated for a range of p, and are displayed in Table 9 . It should be noted that the knock out cannot easily be extended to take advantage of an odd maximum number of games. Table 10 gives the corresponding probabilities for the tournament B(2k) .

In the four player case, B(2k) is merely comparable to the knock out. In the three player case, however, it is noticed that B(2k) is distinctly superior at selecting the outlier in a given maximum number of games. In particular, it should be noted that for $m=8$, 10, and 12 the probability of the outlier winning in B(2k) is uniformly

Table 9

Probability of the Outlier Winning K(k,ℓ)

	(k,ℓ)			
	(3,3)	(3,5)	(5,5)	(5,7)
0.1	.4118	.4249	.4322	.4433
0.2	.4959	.5224	.5382	.5599
0.3	.5833	.6211	.6449	.6747
0.4	.6711	.7164	.7459	.7790
0.5	.7558	.8031	.8346	.8653

Table 10

Probability of the Outlier Winning B(2k)

	m						
	6	7	8	9	10	11	12
0.1	.4125	.4169	.4274	.4309	.4346	.4419	.4447
0.2	.4967	.5053	.5271	.5345	.5418	.5569	.5629
0.3	.5836	.5959	.6281	.6389	.6489	.6705	.6793
0.4	.6705	.6852	.7250	.7383	.7493	.7748	.7846
0.5	.7544	.7698	.8127	.8267	.8370	.8622	.8714

higher than in $K(k,\ell)$ over the range of configurations selected. If an odd maximum number of games were permitted, the knock out must play for one less than the maximum number of games, thus becoming even less efficient than before, relative to BKOT, at choosing the best player.

It has been seen that BKOT with elimination rule $f(k)=2k$ is a reasonable alternative to many types of knock outs and the round robin in the four player case for tournaments of short duration. For the three player case, when the usual knock outs are not expected to be efficient because of byes, BKOT is seen to be definitely preferable to them. In addition, the provision for elimination can be seen to provide considerable flexibility in the design of tournaments of this type. In Chapter III , it will be seen that BKOT can be equipped with an elimination rule which vastly increases its effectiveness in tournaments of moderate duration.

CHAPTER III

TOURNAMENTS OF MODERATE DURATION

In order to compare the "play-the-winner" tournaments with the round robin in the moderate duration case, computer simulations were employed. In this chapter, a new eliminatory scheme is introduced for BKOT in the moderate duration case. In addition, BKOT is modified to compensate for configurations less favourable than the single outlier, and BKOT is compared via simulations with the round robin. Finally, some theoretical results, first announced by T.V. Narayana, are proved which indicate why this design may be expected to outperform the round robin.

In Chapter I it was seen how to compute the smallest number of replications of the round robin required to ensure with at least a predetermined probability P the selection of the single strong outlier. Such values are tabulated by David [4]. From these were obtained the number of games to which the round robin and tournament KOT were to be simulated. Results of simulations for various cases are displayed in Table 11 , For comparative purposes, the results of simulations of the final version of BKOT , called BKOT* , which is defined below, are also included in Table 11 as well as in Table 13 .

Also given in the tables are the means of the "best" subset size in the sense of Gupta and David [4] as

Table 11

Proportion of Wins and Best Sample Size in 100 Simulations

P = .75

(a) p= .55

		RR		KOT		BKOT *	
n	4	.75	2.30	.745	2.21	.84 *	1.32 *
	5	.77	2.76	.83	2.71	.86 *	1.16 *
	6	.748	3.60	.81	3.07	.86 *	1.30 *
	7	.745	3.94	.695	3.53	.90 *	1.50 *
	8	.70	4.05	.74	3.97	.90 *	1.50 *

(b) p = .65

		RR		KOT		BKOT *	
n	4	.74	2.70	.755	2.29	.90 *	1.22 *
	5	.80	2.92	.80	2.47	.84 *	1.46 *
	6	.74	3.27	.808	2.33	.86 *	1.42 *
	7	.722	3.72	.895	3.04	.84 *	1.34 *
	8	.782	4.52	.835	3.25	.80 *	2.10 *

(c) p = .75

		RR		KOT		BKOT *	
n	4	.807	2.53	.77	2.29	.92 *	1.54 *
	5	.793	3.39	.89	2.31	.96 *	1.86 *
	6	.788	3.16	.91	2.3	.88 *	1.34 *
	7	.755	4.43	.89	2.61	.98 *	1.1 *
	8	.905	3.67	.95	1.89	.92 *	1.14 *

* Indicates only 50 simulations

was discussed in Chapter I . It will be remembered that the "best" subset consists of those players for whom

$$(3.1) \qquad S_i \geq S_{max} - \nu$$

where S_i is the score of the i^{th} player, S_{max} is the maximum score, and ν is just large enough so that the outlier is included with probability P . Values of ν are tabulated by David [4].

Although the elimination rules suggested for BKOT in Chapter II are still valid for the moderate duration tournaments, the following scheme is introduced as a much more powerful alternative. A player is eliminated if his score reaches a predetermined value $-k$. A new tournament is then begun among the survivors. The procedure continues until $n-1$ players have been eliminated, whereupon the lone survivor is declared the winner of the tournament. At each game played by the single strong outlier, his score increases by one with probability p and decreases by one with probability q where $q=1-p$. Thus his score describes a simple random walk with absorbing barrier at $-k$. It can be shown (Feller[5]) that the probability of the outlier reaching a "depth" of $-k$ is

$$\left(\frac{q}{p}\right)^k .$$

Since there are $n-1$ eliminations, the outlier will survive the tournament with probability at least

$$(3.2) \qquad \left(1 - \left(\frac{q}{p}\right)^{k}\right)^{n-1}$$

This implies that if k is chosen to be the smallest integer such that (3.2) is at least equal to P for a pre-specified probability P, then the outlier will win the tournament with probability greater than P. Values of k are given in Table 12 which was derived by Brett [3].

The form of BKOT used in the simulations varied slightly from this basic form. By beginning a new tournament after each elimination, much information is lost. In particular, the outlier loses an advantage since he has a relatively high probability of having an above average score. Thus for BKOT the scores were retained after each elimination while the level $-k$ was raised by an amount $(n-1)/k$ to "effectively" adjust the scores so as to have zero mean. This should obviously increase the probability of the outlier avoiding elimination and thus winning the tournament.

To provide a meaningful comparison with the round robin, the procedure was truncated after the same maximum number of games as was used in Table 11, whether or not there was a lone survivor.

The players uneliminated at termination form the "best" subset for this procedure. In cases where more than one player remained, the player with the highest score was

Table 12

Smallest k Such That $\left(1-\left(\frac{q}{p}\right)^{k}\right)^{n-1} \geq P$

(a) p = .55

		n 2	3	4	5	6	7	8	9	10
P	.75	7	11	12	14	15	16	17	17	18
	.90	12	15	17	19	20	21	21	22	23
	.95	15	19	21	22	23	24	25	26	26
	.99	23	27	29	30	31	32	33	34	34

(b) p = .65

		n 2	3	4	5	6	7	8	9	10
P	.75	3	4	4	5	5	5	6	6	6
	.90	4	5	6	6	7	7	7	8	8
	.95	5	6	7	8	8	8	8	9	9
	.99	8	9	10	10	11	11	11	11	11

(c) p = .75

		n 2	3	4	5	6	7	8	9	10
P	.75	2	2	3	3	3	3	3	4	4
	.90	3	3	4	4	4	4	4	4	5
	.95	3	4	4	4	5	5	5	5	5
	.99	5	5	6	6	6	6	6	7	7

declared winner of the tournament. In case of a tied score, the player with the least total plays was chosen. In order to break possible further deadlocks, a tie-breaking by fiat as in Chapter II was employed.

One further modification was made to BKOT in anticipation of the arguements of Chapter IV with respect to elimination of weak players in configurations other than that of the strong outlier. Until the first elimination, the order of play is reversed, that is, the loser plays through and a player of least score begins each replication. After the first elimination, or when half the allowable number of games have been played, if no elimination has occurred by then, the order of play reverts to the conventional BKOT for the remainder of the tournament. It will be seen in Chapter IV , that a test may be made for the equality of scores, the result of which may be used to govern the procedure involked for more general configurations.

Using the computer programme listed in Appendix II, this BKOT procedure, denoted BKOT* , was simulated using the appropriate levels -k . Results are displayed for various cases for P*=.75 in Table 13 , and also for P*=.90 in Table 14 .

It is immediately clear that BKOT* seems far superior to the round robin in the sense of choosing the strong player. In addition, the size of the "best" subset

Table 13

Results of 50 Simulations of BKOT* , P=.75

	n	Subset	Success	Saving	m
p=.55	3	1.32	.68	59.24	207
	4	1.16	.84	172.40	426
	5	1.28	.86	198.28	680
	6	1.30	.86	276.86	975
	7	1.50	.90	211.54	1281
	8	1.50	.90	313.2	1650
p=.65	3	1.44	.80	6.22	24
	4	1.22	.90	14.62	48
	5	1.46	.84	16.94	80
	6	1.42	.86	20.00	105
	7	1.34	.84	39.28	147
	8	2.10	.80	29.20	196
p=.75	3	1.18	.88	3.26	9
	4	1.54	.92	3.36	18
	5	1.86	.96	4.06	30
	6	1.34	.88	12.68	45
	7	1.10	.98	24.92	63
	8	1.14	.92	27.68	84

(Part of Table 13 is reproduced in Table 11)

Table 14

Results of 50 Simulations of BKOT* , P=.90

	n	Subset	Success	Saving	m
p=.55	3	1.12	.92	253.54	495
	4	1.10	.92	396.26	900
	5	1.16	.98	528.78	1350
	6	1.08	.98	766.14	1830
	7	1.14	.94	769.26	2352
	8	1.02	.98	1006.98	2884
p=.65	3	1.02	.92	24.68	54
	4	1.06	.94	44.78	96
	5	1.04	.94	67.72	150
	6	1.12	.96	57.80	195
	7	1.02	.98	91.32	252
	8	1.10	1.00	114.82	308
p=.75	3	1.14	.92	6.24	18
	4	1.32	.98	10.34	36
	5	1.04	.98	18.36	50
	6	1.16	1.00	24.68	75
	7	1.14	1.00	28.84	84
	8	1.32	.92	40.72	112

is drastically reduced, and furthermore, a considerable saving in games is realized. This can be explained by noting the increased proportion of games played by the outlier in KOT and BKOT as compared to the round robin. Of course, the proportion of games which every player, including the outlier, plays in a round robin is $2/n$. In order to obtain the proportion for KOT, consider every replication rather than a play as the basic unit. Then KOT becomes a Markov chain if the following definition is made.

Definition. In playing KOT, let the system be said to be in state i in any replication, $(i=1,2,\ldots,n-1)$ if the strong outlier first entered the current replication in game i.

Then in the initial replication, using the notation of Feller [5],

$$a_1=2/n \ , \quad a_2=\ldots=a_{n-1}=1/n$$

where a_i is the probability the system is in state i.

Theorem 3.1 The proportion of replications initiated by the strong outlier in KOT approaches

$$\frac{1}{q(n-2)+1}$$

Proof. The transition probabilities p_{ij} from replication ν to $\nu+1$, $(\nu\geq 1)$ are

$$p_{11} = p^{n-1} + \frac{1-p^{n-1}}{n-1}$$

$$p_{ij} = \frac{1-p^{n-1}}{n-1} \quad , \ (j \ge 2)$$

for all $i=1,2,\ldots,n-1$.

It is a straightforward calculation to obtain the stationary distribution v from $vP = v$. Indeed noting $v_2 = \ldots = v_{n-1}$, it follows that

$$v_1 = \frac{1}{(n-2)q + 1}$$

$$v_j = \frac{q}{(n-2)q + 1} \quad , \ (j \ge 2)$$

The theorem is proved.

Since $v_1 > v_2 = v_3 = \ldots = v_{n-1}$, it is clear that asymptotically, the strong player plays more games in KOT than in the round robin. Furthermore, interchanging the role of winner and loser, and denoting this tournament by TOK , the outlier has probability q of "winning" the game. Theorem 3.1 states he plays a proportion

$$\frac{1}{(n-2)p + 1}$$

of games in the initial position, and only

$$\frac{p}{(n-2)p + 1}$$

of games in the 2^{nd} , 3^{rd} ,... positions. From these remarks follows the corollary below.

Corollary. The proportion of replications initiated by the strong outlier in the tournament consisting of equal numbers of TOK and KOT replications approaches asymptotically a quantity greater than $2/n$.

Proof. The proportion in question is

$$P = \frac{1}{2}\left\{\frac{1}{p(n-2)+1} + \frac{1}{q(n-2)+1}\right\}$$

$$= \frac{(q+p)(n-2)+2}{2[pq(n-2)^2 + (n-2)+1]}$$

$$= \frac{n}{2[pq(n-2)^2 + (n-2)+1]}$$

$P(p)$ is clearly minimized when $p = q = 1/2$, in which case

$$P(1/2) = \frac{n/2}{\frac{(n-2)^2}{4} + (n-2) + 1} = 2/n \quad .$$

Finally, in BKOT the player with the highest score initiates each replication. Since the strong outlier eventually obtains the highest score, asymptotically he will initiate all replications with probability 1 , which yields the following result.

Theorem 3.2 Asymptotically the proportion of games played by the strong outlier in BKOT approaches $(1-p^{n-1})/q(n-1)$.

Proof. Let X be the number of games eventually played by the outlier in one replication.

Noting $Pr(X=i)=qp^{i-1}$, $(i=1,2,\ldots,n-2)$

and $\Pr(X=n-1)=p^{n-2}$,

It can be seen that

$$E(X)=\frac{1-p^{n-1}}{q}$$

This implies the result.

Similarly, the same reasoning may be extended to provide the corresponding result for KOT .

Theorem 3.3 Asymptotically, the proportion of games played by the strong outlier in KOT approaches

$$\frac{1}{(n-2)q+1} \quad .$$

Proof: Using the same notation as in Theorem 3.2 , note

$$E(X)=\frac{1}{(n-2)q+1}\Big[(n-1)p^{n-2}+\sum_{j=1}^{n-2}(n-1-j)p^{n-1-j}q$$

$$+q\sum_{i=2}^{n-1}\Big\{(n-i)p^{n-i-1}+\sum_{j=1}^{n-i-1}(n-i-j)p^{n-i-j-1}q\Big\}\Big]$$

which can be reduced to

$$\frac{n-1}{(n-2)q+1}$$

which implies the result.

It can thus be seen that in KOT and BKOT , a larger proportion of the games involve the outlier than is the case with the round robin. In moderately long tournaments, it can therefore be expected that the outlier will build a higher score and therefore win more often, as in implied by the theory and borne out by the extensive simulations. The reduced "best" subset size, and the saving of games also contributes to the superiority of BKOT over the round robin.

CHAPTER IV

ASPECTS OF TOURNAMENT DESIGN

In Chapter III the tournament BKOT was developed and the design BKOT* chosen to compare, via simulation, with the round robin. Although this sort of design is obviously to some extent arbitrary, there are nevertheless definite motives for the choice. In this chapter some of these concepts of tournament design are introduced, and suggestions are made for further improvements to the basic BKOT design.

Initially, it should be noted that, in general, a good design is one which maximizes the proportion of games played by the strongest player. This is carefully discussed by Narayana and Zidek [9] and forms the rationale behind the original KOT and BKOT designs. Tables 15 and 16 give the asymptotic proportions of play for the strong outlier computed with the help of Theorems 3.2 and 3.3 . Table 17 gives the expected proportion of play for the outlier in the replicated classical knock out. Table 18 , by way of comparison with the results of Chapter III , gives the probability of the outlier winning the extended knock out based on the maximum number of games computed from the tables mentioned in Chapter I , and where the probabilities are computed using standard binomial

Table 15

Expected Proportion of Play for Outlier (Asymptotic)-KOT

		p		
		.55	.65	.75
n	3	.69	.74	.80
	4	.53	.59	.67
	5	.43	.49	.57
	6	.36	.42	.50
	7	.31	.36	.44
	8	.27	.32	.40

Table 16

Expected Proportion of Play for Outlier (Asymptotic)-BKOT

		p		
		.55	.65	.75
n	3	.78	.83	.88
	4	.62	.69	.77
	5	.50	.59	.68
	6	.42	.51	.61
	7	.36	.44	.55
	8	.32	.39	.50

Table 17

Expected Proportion of Play for Outlier -Classical Knock Out

	p		
n	.55	.65	.75
3	.69	.72	.75
4	.52	.55	.58
5	.42	.46	.49
6	.35	.39	.42
7	.30	.34	.37
8	.26	.29	.33

Table 18

Probability of Outlier Winning Extended Knock Out

P=.75

	p		
n	.55	.65	.75
3	.739	.796	.834
4	.766	.812	.863
5	.759	.826	.887
6	.781	.808	.875
7	.793	.838	.906
8	.810	.863	.930

(The values in Table 18 should be considered as upper bounds as interpolation of tables [14] was not employed.)

tables [14], and with reference to Glenn [6]. Finally, Table 19 shows the actual proportion of games played by the outlier in the simulations mentioned in Chapter III. It can easily be seen that even with no elimination, the outlier plays more often in BKOT than in the knock out, whereas the simulations indicate that the outlier plays significantly more in $BKOT^*$ than in the knock out.

It is reasonable to suppose that the elimination of a player other than the outlier will increase the outlier's proportion of plays. To this end it is important to inspect the design of the elimination procedure. Suppose it is desired to eliminate from competition any player whose score drops to a pre-specified value $-k$. This process is to be repeated until $n-1$ players have been eliminated, whereupon the survivor is to be called the winner. It was seen in the preceeding chapter that the outlier's score describes a random walk, and it can be seen (Feller [5]) that the probability of the outlier's score first falling to $-k$ is less than

$$\left(\frac{q}{p}\right)^k .$$

Three possible courses of action to be followed after a player is eliminated are considered:

(A) Upon the elimination of one of n players, the scores of the remainder are returned to zero and a new tourn-

Table 19

Proportion of Play for Outlier in 50 Simulations -BKOT*

P = .75

	p		
n	.55	.65	.75
3	.73	.73	.73
4	.66	.66	.68
5	.58	.58	.60
6	.51	.55	.60
7	.48	.49	.58
8	.45	.43	.50

ament is begun among the $n-1$ survivors. This process is repeated until a single player is left. The probability of the outlier being the first to reach $-k$ is obviously less than

$$\left(\frac{q}{p}\right)^k .$$

This relation holds for all $n-1$ eliminations, so the probability of the outlier winning is greater than

$$\left(1 - \left(\frac{q}{p}\right)^k\right)^{n-1} .$$

A given level of probability P can thus be attained by appropriate choice of k -- see Table 12 .

(B) Upon elimination, the scores made against the eliminated player by the surviving players are removed from the survivors' scores. The survivors retain, however, the scores made among themselves, and continue to play until only the winner remains. Suppose the original scores of the survivors are $(S_1, S_2, \ldots, S_{n-1})$ and the corrected scores $(S'_1, S'_2, \ldots, S'_{n-1})$; note $\sum S_i = k$, but $\sum S'_i = 0$. If the first player is the strong outlier, then $E(S'_1) > 0$ and hence the probability of the strong player descending to $-k$ should be less than

$$\left(\frac{q}{p}\right)^k ,$$

provided he is not already eliminated. Thus it is reasonable to claim that (B) will enable the outlier to win with a higher probability than will (A) .

(C) All scores are retained, and corrections made simply by the relation $S'_i = S_i - k/(n-1)$, so that once again $\sum S'_i = 0$. Clearly, (C) exhibits the same advantages mentioned for (B) . In addition, the outlier presumably will have won against the eliminated player with a higher probability than the other survivors, and hence (C) can be expected to perform slightly better than (B) .

The choice among these methods is determined largely by intuitive rather than by rigorous mathematical arguments. Method (A) uses strict independence of the eliminations and to that extent is mathematically exact; however, at every elimination a large amount of useful information is discarded; in particular, the results of games among the survivors are ignored. This is undesirable since the strong player has an expected positive score against all his opponents which would be returned to zero after each elimination. Methods (B) and (C) retain most of this information, with method (C) using all available data, and to this extent is superior to the other methods. Results of simulations definitely support this view, but no theoretical resuls are available. Because the strict independence of method (A) does not hold for (B) and (C) these two should be considered as stochastic approximation methods. Since BKOT is partially balanced, it is not expected that the lack of independence in methods (B) and (C) will have a serious effect when the pre-assigned probability P is close to 1 .

From practical considerations it is desirable to terminate the tournament after a pre-specified maximum number of games, even if fewer than n-1 players have been eliminated. In the simulations of Chapter III for example, the maximum number of games was that corresponding to the number of replications required for the round robin to choose the best player with a given probability, as described in Chapter I . In cases where there is more than one survivor, of course the survivor with the highest score is declared the winner. In actual simulations it has been found that in only a small proportion of cases does the tournament require the maximum number of games without eliminating all but one player.

As was mentioned in Chapter III , there exists a tournament "dual" to BKOT , which may be denoted BTOK . Essentially, BTOK is played under the same rules as is BKOT except that the loser, rather than the winner, plays in each succeeding game, and a player with least, rather than greatest, score begins each replication. It is clear that the theory concerning the elimination level -k applies equally for BTOK as for BKOT . It is with these considerations in mind that the tournament BKOT* is defined as follows. Play begins according to the rules of BTOK with maximum number of games to be played in this fashion being one half of the total. Upon elimination of any player during BTOK , the play continues under the rules of BKOT for the duration of the tournament. If no elimination occurs by the time that one half the maximum number of games have been played, play continues under the rules of BKOT from that point. In any case, method (C) of adjusting the scores is employed at every elimination.

The purpose of the preliminary application of BTOK is to attempt to avoid any possible problems with a configuration less favourable than the single outlier situation. It is pointed out in Chapter I that the single outlier case, although extremely important in its own right, does not represent the least favourable configuration for the purpose of selecting the best player. To quote an example from David [4] , consider one replication of a five player round robin with single outlier, the five players being augmented by fifteen totally worthless players. The outlier's probability of winning is identical with that in the simple five player case which is less than that in the simple twenty player outlier case for all situations where the probability of the outlier winning any one game lies in the interval (.69 , .95) .

It is hoped that by means of BTOK , poor players may be quickly eliminated, and thus configurations less favourable than the single outlier situation will be quickly reduced to configurations approximating that of the single outlier. In practice, a test might be made at each elimination to determine whether or not a congenial configuration might be reasonably assumed. Depending on the outcome of this test, play would continue according to the rules of BTOK or BKOT . To this end a test for the equality of scores is in order.

For the n player round robin of t replications it has been shown (Starks and David [12] , David [4]) that the statistic

$$D_t = 4[\sum_{i=1}^{n} a_i^2 - (1/4)nt^2 (n-1)^2]/tn \quad ,$$

where a_i is the number of wins for player i , and has a limiting χ^2_{n-1} distribution as $t \to \infty$. Since the asymptotic results of [9] indicate that this also holds for BKOT and BTOK , a statistic can be used to test for the equality of the players merely by altering the scoring as follows. Given the scores $(S_1, S_2, \ldots, S_n)$, it is seen that

$$S_i = (\text{number of wins for } i) - (\text{number of losses})$$

$$= a_i - [t(n-1)-a_i] \quad ;$$

thus $$a_i = \frac{S_i + t(n-1)}{2} \quad , \text{ and}$$

clearly

$$D_t = \frac{\sum_{i=1}^{n} s_i^2}{tn}$$

is the corresponding more general statistic, which can be used for knock outs or round robins.

With the aid of the D_t statistic, it is now possible to test for the equality of scores at any point. Since the single outlier case represents a relative equality of strength among players, a significant test result would imply the existence of players who were relatively weak. In such a case BTOK is prefered to BKOT because of its ability to eliminate the weak player quickly. If the test is made at the time of each elimination, the following general procedure is suggested: If, at any elimination, the test does not prove significant, play continues according to BKOT, method (C) being used to adjust the scores. As long as the test proves significant, the scores of all but the last eliminated player are discarded, and play continues according to BTOK. It is thus assured that all worthless or near worthless players are eliminated as quickly as possible by the repeated use of BTOK; while, at the same time, discarding the scores of previously eliminated players when there is significance precludes the possibility of any player building up a high score at the expense of the weak or worthless players. A non-significant test is taken as indicating a near outlier configuration in which case it has been seen that BKOT is highly effective.

This method of discarding the scores of all previously eliminated players can also be useful if a ranking of the players is desired. A natural ranking system is to be had if the players are ranked in the inverse order of their elimination. However, it can easily be seen that by treating the scores with method (C) , it is quite possible that more than one player may be eliminated after the same game. By discarding scores, the eliminations are spread out so that an unambiguous ranking can be made.

Finally, something should be said about the practice of changing to BKOT after one half the maximum number of games regardless of tests. This is admittedly somewhat arbitrary, but intuitively it may be thought of as compensating for the probability that no eliminations have been made by that time. Simulations have been done to some extent when the change to BKOT is made after 25% and 95% of the maximum possible number of games, and no distinct pattern has emerged, although the 50% changeover seems best.

In passing, it should be noted that the suggested BKOT design, although valid for tournaments of any number of players, cannot be expected to show spectacular improvements over other BKOT designs in the three player case, since after the first elimination, BTOK and BKOT are identical. The three player case is worthy of further study, especially as an exact investigation appears possible.

In conclusion, it is strongly suggested that a tournament based on BKOT* is the most efficient yet devised for determining the strongest player. In addition, BKOT* offers significant improvement in choosing a subset containing the strongest player and also exhibits an economy in the number of games required. It is regretted that there are gaps in the underlying mathematical theory, especially the lack of a least favourable configuration for the preference probabilities with respect to selecting the best player. However, BKOT* is extremely advantageous to use when variations from the one outlier case occur, since BTOK makes for rapid elimination; while in the case of a configuration approximating that of the single outlier, BKOT* quickly reduces to BKOT which is known to be highly effective. It is thus no surprise that numerical comparisons in the short duration case, asymptotic results on the frequency of play, and extensive simulations all support the view that the BKOT* type of tournament is superior to the round robin and other common tournament types. A decision scheme based on tests is suggested to further increase the effectiveness of BKOT*; even without this increased effectiveness it is clear that BKOT* as suggested exhibits the advantages of both round robins and pure knock outs, without any of their disadvantages. Thus the spectacular reduction in subset size achieved, as well as the accuracy in choosing the best player, is due to choosing a good design which incorporates all advantages, but not the defects of particular designs.

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APPENDIX I

```
C
C
C          A PROGRAMME TO COMPUTE PARAMETERS OF  BKOT
C
C
C    THE INPUT CONSISTS OF THE MAXIMUM NUMBER OF GAMES AND
C    THE NUMBER OF PLAYERS, AS WELL AS AN ARBITRARY NUMBER
C    OF CONFIGURATIONS OF PREFERENCE  PROBABILITIES.
C
C    THE ELIMINATION RULE (HERE F(K)=2K) MAY BE CHANGED BY
C    ALTERING STATEMENT  300 .
C
C    THE OUTPUT CONSISTS OF THE EXPECTED NUMBER OF GAMES, AS
C    WELL AS THE PROBABILITY OF EACH OF THE PLAYERS WINNING,
C    FOR EACH CONFIGURATION.
C
C
      REAL TIE(20)
      REAL EXP(20)
      REAL PROB(9,9,20), PWIN(9,20), RPROB(12,20)
      INTEGER STACK(12,36,2), PLAYM(9,9), PLAYV(9)
      INTEGER PRAND(12), SCORE(9)
      INTEGER MSTACK(12), REP(12), WIN,LOSE,NSTACK
      LOGICAL CAND(9), LASTW(9,12), LASTL(9,12), ELIM(9,12)
      LOGICAL FLAG
10    READ(5,200,END=211)M,N,NO
      DO 5 IN=1,NO
      TIE(IN)=0.
      EXP(IN)=0.
      DO 5 I=1,N
      PWIN(I,IN)=0
5     READ(5,201)(PROB(I,J,IN),J=1,N)
      DO 2 I=1,N
      DO 1 J=1,N
      PLAYM(I,J)=0
1     CONTINUE
      PLAYV(I)=0
      SCORE(I)=0
2     CONTINUE
      DO 4 I=1,M
4     REP(I)=N-1+(N-1)*IFIX(FLOAT((I-1)/(N-1)))
C
C    INITIALIZE
C
      DO 100 II=1,N
      DO 100 JJ=1,N
      IF(II.EQ.JJ)GO TO 100
      DO 3 I=1,N
      CAND(I)=.TRUE.
      DO 3 J=1,M
      LASTW(I,J)=.FALSE.
      LASTL(I,J)=.FALSE.
      MSTACK(J)=0
    3 ELIM(I,J)=.TRUE.
      WIN=II
      LOSE=JJ
      PLAYM(II,JJ)=1
      PLAYM(JJ,II)=1
      PLAYV(II)=1
      PLAYV(JJ)=1
      SCORE(II)=1
      SCORE(JJ)=-1
      CAND(II)=.FALSE.
      NSTACK=1
      LASTW(WIN,1)=.TRUE.
      LASTL(LOSE,1)=.TRUE.
      DO 11 IN=1,NO
11    RPROB(1,IN)=2.*PROB(WIN,LOSE,IN)/(N*(N-1))
C
C    PUSH DOWN
C
30    IF(NSTACK.EQ.M)GO TO 60
      DO 40 I=1,N
      DO 40 J=1,N
      IF(I.EQ.WIN)GO TO 40
300   IF((SCORE(J)-SCORE(I)).LT.(2*(M-NSTACK)   ))GO TO 40
      ELIM(I,NSTACK)=.FALSE.
      CAND(I)=.FALSE.
40    CONTINUE
      K=0
      DO 41 I=1,N
      IF(.NOT.ELIM(I,NSTACK))GO TO 41
      K=K+1
41    CONTINUE
      IF(K.EQ.1)GO TO 60
      L=0
      DO 44 IJ=1,N
      DO 43 I=1,NSTACK
      IF((LASTW(IJ,I).OR.LASTL(IJ,I).OR.
     1(.NOT.ELIM(IJ,NSTACK))).AND.
     1(REP(I).GE.REP(NSTACK)))GO TO 44
```

```
43    CONTINUE
      L=L+1
44    CONTINUE
      J=M-L
      DO 42 I=NSTACK,J
42    REP(I+L)=(K-1)*IFIX(FLOAT((I+K-NSTACK-2)/(K-1)))
     1+NSTACK+L
      IF(REP(NSTACK).EQ.NSTACK)GO TO 12
      DO 31 IJ=1,N
      DO 31 I=1,NSTACK
      IF((LASTW(IJ,I).OR.LASTL(IJ,I).OR.
     1(.NOT.ELIM(IJ,NSTACK))).AND.
     1(REP(I).GE.REP(NSTACK)))CAND(IJ)=.FALSE.
31    CONTINUE
      GO TO 35
12    CONTINUE
      DO 32 I=1,N
      IF(SCORE(I).GT.SCORE(WIN))GO TO 33
32    CONTINUE
      GO TO 35
33    DO 34 I=1,N
      DO 34 J=1,N
      IF(SCORE(I).LT.SCORE(J))CAND(I)=.FALSE.
34    CONTINUE
35    CONTINUE
      DO 36 I=1,N
      DO 36 J=1,N
      IF(PLAYM(WIN,I).GT.PLAYM(WIN,J).AND.CAND(J))
     1CAND(I)=.FALSE.
36    CONTINUE
      DO 37 I=1,N
      DO 37 J=1,N
      IF(PLAYV(I).GT.PLAYV(J).AND.CAND(J))CAND(I)=.FALSE.
37    CONTINUE
      NSTACK=NSTACK+1
      K=0
      DO 38 I=1,N
      IF(.NOT.CAND(I))GO TO 38
      STACK(NSTACK,K+1,1)=WIN
      STACK(NSTACK,K+1,2)=I
      STACK(NSTACK,K+2,1)=I
      STACK(NSTACK,K+2,2)=WIN
      K=K+2
38    CONTINUE
      PRAND(NSTACK)=K/2
      MSTACK(NSTACK)=1
```

```
C
C  UPDATE
C
29    CONTINUE
      DO 28 I=1,N
      ELIM(I,NSTACK)=ELIM(I,NSTACK-1)
28    CAND(I)=ELIM(I,NSTACK)
      WIN=STACK(NSTACK,MSTACK(NSTACK),1)
      LOSE=STACK(NSTACK,MSTACK(NSTACK),2)
      LASTW(WIN,NSTACK)=.TRUE.
      LASTL(LOSE,NSTACK)=.TRUE.
      SCORE(WIN)=SCORE(WIN)+1
      SCORE(LOSE)=SCORE(LOSE)-1
      PLAYM(WIN,LOSE)=PLAYM(WIN,LOSE)+1
      PLAYM(LOSE,WIN)=PLAYM(LOSE,WIN)+1
      PLAYV(WIN)=PLAYV(WIN)+1
      PLAYV(LOSE)=PLAYV(LOSE)+1
      CAND(WIN)=.FALSE.
      DO 50 IN=1,NO
50    RPROB(NSTACK ,IN)=RPROB(NSTACK-1,IN)*PROB
     1(WIN,LOSE,IN)
     C/PRAND(NSTACK)
      GO TO 30
C
C    POP UP
C
60    CONTINUE
      DO 61 I=1,N
61    CAND(I)=ELIM(I,NSTACK)
      DO 62 I=1,N
      DO 62 J=1,N
      IF(SCORE(I).LT.SCORE(J).AND.CAND(J))CAND(I)=.FALSE.
62    CONTINUE
      FLAG=.FALSE.
      DO 63 I=1,N
      DO 63 J=1,N
      IF(I.EQ.J)GO TO 63
      IF(SCORE(I).NE.SCORE(J))GO TO 63
      IF((.NOT.CAND(I)).OR. (.NOT.CAND(J)))GO TO 63
      IF(FLAG)GO TO 69
      FLAG=.TRUE.
      DO 68 IN=1,NO
68    TIE(IN)=TIE(IN)+RPROB(NSTACK,IN)
69    CONTINUE
      DO 64 K=1,M
      IF(LASTW(I,M+1-K).AND.(.NOT.LASTW(J,M+1-K)))
     1CAND(J)=.FALSE.
      IF((.NOT.LASTW(I,M+1-K)).AND.LASTW(J,M+1-K))
     1CAND(I)=.FALSE.
      IF((.NOT.CAND(I)).OR.(.NOT.CAND(J)))GO TO 63
 64   CONTINUE
 63   CONTINUE
      DO 70 I=1,N
      IF(.NOT.CAND(I))GO TO 70
      DO 55 IN=1,NO
      PWIN(I,IN)=PWIN(I,IN)+RPROB(NSTACK,IN)
      EXP(IN)=EXP(IN)+RPROB(NSTACK,IN)*NSTACK
 55   CONTINUE
 70      CONTINUE
 67   CONTINUE
      DO 65 I=1,N
      ELIM(I,NSTACK)=.TRUE.
 65   CAND(I)=ELIM(I,NSTACK-1)
      WIN=STACK(NSTACK,MSTACK(NSTACK),1)
      LOSE=STACK(NSTACK,MSTACK(NSTACK),2)
      LASTW(WIN ,NSTACK)=.FALSE.
      LASTL(LOSE,NSTACK)=.FALSE.
      SCORE(WIN)=SCORE(WIN)-1
      SCORE(LOSE)=SCORE(LOSE)+1
      PLAYM(WIN,LOSE)=PLAYM(WIN,LOSE)-1
      PLAYM(LOSE,WIN)=PLAYM(LOSE,WIN)-1
      PLAYV(WIN)=PLAYV(WIN)-1
      PLAYV(LOSE)=PLAYV(LOSE)-1
      IF(MSTACK(NSTACK).EQ.2*PRAND(NSTACK))GO TO 66
      MSTACK(NSTACK)=MSTACK(NSTACK)+1
      GO TO 29
 66   NSTACK=NSTACK-1
      IF(NSTACK.GT.1)GO TO 67
      PLAYM(II,JJ)=0
      PLAYM(JJ,II)=0
      PLAYV(II)=0
      PLAYV(JJ)=0
      SCORE(II)=0
      SCORE(JJ)=0
      CAND(II)=.TRUE.
      LASTW(II,1)=.FALSE.
      LASTL(JJ,1)=.FALSE.
100   CONTINUE
      DO 51 IN=1,NO
      WRITE(6,220)EXP(IN)
      WRITE(6,215) TIE(IN)
      WRITE(6,205)(PWIN(I,IN), I=1,N)
      DO 51 I=1,N
      WRITE(6,202)(PROB(I,J,IN),J=1,N)
51    CONTINUE
      GO TO 10
211   CONTINUE
200   FORMAT(6X,3I3)
201   FORMAT(' ',9F6.4)
202   FORMAT(' ',9(3X,F6.4))
205   FORMAT('0','PROB OF WINNING  ',9(3X,F6.4)/)
210   FORMAT(' ',I3,5X,9(3X,2I3))
215   FORMAT('0PROBABILITY OF A TIE ',F6.4)
220   FORMAT(///'0EXPECTED NUMBER OF GAMES ',F7.4)
      STOP
      END
```

APPENDIX II

```
C
C
C              A PROGRAMME TO SIMULATE BKOT*
C
C
C   THE INPUT CONSISTS OF THE FOLLOWING:
C  (1)  THE INITIAL VALUE OF THE RANDOM NUMBER GENERATOR
C       AS OBTAINED FROM THE PREVIOUS RUN OF THE PROGRAMME.
C  (2)  THE NUMBER OF SIMULATIONS TO BE MADE.
C  (3)  THE ELIMINATION LEVEL.
C  (4)  THE NUMBER OF PLAYERS AND THE MAXIMUM NUMBER OF
C       GAMES.
C  (5)  THE MATRIX OF PREFERENCE PROBABILITIES.
C
C  THE OUTPUT CONSISTS OF A SUMMARY OF EACH ELIMINATION,
C  INCLUDING THE SCORES AND NUMBER OF PLAYS FOR EACH
C  PLAYER, THE WINNER AND DURATION OF THE TOURNAMENT,
C  AND A SUMMARY OF THE FULL SET OF SIMULATIONS ,
C  INCLUDING THE INITIAL VALUE FOR THE RANDOM NUMBERS
C  IN THE NEXT RUN.
C
C  THIS PROGRAMME REQUIRES SUBROUTINE  URAND  FROM  CSLIB
C
      REAL PROB(10,10)
      INTEGER WINM(10,10),PLAYM(10,10),SCORE(10),LASTW(10)
      INTEGER WIN,LOSE,REP,PLAYV(10),LASTP(10)
      LOGICAL ELIM(10),REJT(10)
      REAL LIMJ
C
C   INITIALIZATION
C
      READ(5,200)INIT
      READ(5,200)MANY
      READ(5,200)LIM
      READ(5,201)N,M
      DO 7 I=1,N
      READ(5,202)(PROB(I,J),J=1,N)
7     CONTINUE
      XS=0.
      XW=0.
      XL=0.
      DO 100 IDO=1,MANY
      LIMJ=LIM
      IFLG=1
      WRITE(6,218)N,M
      WRITE(6,210)INIT
      DO 2 I=1,N
      DO 1 J=1,N
      WINM(I,J)=0
1     PLAYM(I,J)=0
      SCORE(I)=0
      LASTP(I)=0
      ELIM(I)=.FALSE.
      REJT(I)=.FALSE.
      PLAYV(I)=0
2     LASTW(I)=0
      NO=0
      NLFT=N
      REP=1
      X=URAND(INIT)
      J=INT(1000.*X)
      IA=MOD(J,N)+1
3     X=URAND(INIT)
      J=INT(1000.*X)
      IB=MOD(J,N)+1
      IF(IA.EQ.IB)GO TO 3
6     CONTINUE
      IF(REP+NLFT-2.GT.(M/2))IFLG=0
      IF(PROB(IA,IB).LT.X)GO TO 4
      WIN =IB*IFLG+IA*(1-IFLG)
      LOSE=IA*IFLG+IB*(1-IFLG)
      GO TO 5
4     WIN =IA*IFLG+IB*(1-IFLG)
      LOSE=IB*IFLG+IA*(1-IFLG)
5     CONTINUE
C
C   UPDATE
C
      NO=NO+1
      PLAYM(WIN,LOSE)=PLAYM(WIN,LOSE)+1
      PLAYM(LOSE,WIN)=PLAYM(LOSE,WIN)+1
      WINM(WIN,LOSE)=WINM(WIN,LOSE)+1
      PLAYV(WIN)=PLAYV(WIN)+1
      PLAYV(LOSE)=PLAYV(LOSE)+1
      SCORE(WIN )=SCORE(WIN )+1-2*IFLG
      SCORE(LOSE)=SCORE(LOSE)-1+2*IFLG
      LASTW(WIN)=NO
      LASTP(WIN)=NO
      LASTP(LOSE)=NO
      IF(NO.EQ.M)GO TO 40
C
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C     ELIMINATION
C
9         CONTINUE
          DO 10 I=1,N
          IF((SCORE(I).GT.LIMJ).OR.ELIM(I))GO TO 10
          ELIM(I)=.TRUE.
          REJT(I)=.TRUE.
          NLFT=NLFT-1
          IF(IFLG.EQ.1)WIN=LOSE
          IFLG=0
          WRITE(6,219)(SCORE(J),J=1,N)
          WRITE(6,220)(PLAYV(J),J=1,N)
          WRITE(6,217)I,NO
          IF(NLFT.EQ.1)GO TO 40
          LIMJ=LIMJ-LIMJ/NLFT
          GO TO 9
10        CONTINUE
          IF((IFLG.EQ.1).AND.(REP+NLFT-2.GE.(M/2)).AND.
         1 (REP+NLFT-2.EQ.NO)) WIN=LOSE
C
C     CHOOSING FIRST PLAYER AFTER ELIM
C
          IF(.NOT.ELIM(WIN))GO TO 50
          DO 51 I=1,N
51        REJT(I)=ELIM(I)
          J=NLFT
          DO 52 I=1,N
          DO 52 K=1,N
          IF((SCORE(I).LE.SCORE(K)).OR.REJT(K).OR.REJT(I))
         1 GO TO 52
          REJT(I)=.TRUE.
          J=J-1
52        CONTINUE
          X=URAND(INIT)
          L=INIT/2
          J=MOD(L,J)
          K=0
          DO 53 I=1,N
          IF(REJT(I))GO TO 53
          IF(K.EQ.J)IA=I
          K=K+1
53        CONTINUE
          DO 54 I=1,N
54        REJT(I)=ELIM(I)
          REJT(IA)=.TRUE.
          J=NLFT-1
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      GO TO 30
 50   CONTINUE
C
C   REPLICATION COUNT
C
      L=0
      DO 11 I=1,N
      DO 12 J=REP,NO
      IF((LASTP(I).GE.J).OR.ELIM(I))GO TO 18
      GO TO 12
 18   REJT(I)=.TRUE.
      GO TO 11
 12   CONTINUE
      L=L+1
 11   CONTINUE
      IF(L.NE.0)GO TO 20
      DO 14 I=1,N
 14   REJT(I)=ELIM(I)
      REJT(WIN)=.TRUE.
      REP=NO+1
C
C   CHOOSING OF NEXT OPPONENT
C
      DO 15 I=1,N
      IF(SCORE(WIN).LT.SCORE(I))GO TO 16
 15   CONTINUE
      GO TO 20
 16   CONTINUE
      DO 17 I=1,N
      DO 17 J=1,N
      IF(IFLG.EG.0)GO TO 29
      IF((SCORE(I).GT.SCORE(J)).AND.(.NOT.REJT(J)))
     1 REJT(I)=.TRUE.
      GO TO 17
 29   IF((SCORE(I).LT.SCORE(J)).AND.(.NOT.REJT(J)))
     1 REJT(I)=.TRUE.
 17   CONTINUE
 20   CONTINUE
      DO 21 I=1,N
      DO 21 J=1,N
      IF((PLAYM(WIN,I).GT.PLAYM(WIN,J)).AND.
     1 (.NOT.REJT(J)))REJT(I)=.TRUE.
 21   CONTINUE
      DO 22 I=1,N
      DO 22 J=1,N
      IF((PLAYV(I).GT.PLAYV(J)).AND.(.NOT.REJT(J)))
     1 REJT(I)=.TRUE.
   22  CONTINUE
       IA=WIN
       J=0
       DO 23 I=1,N
       IF(.NOT.REJT(I))J=J+1
   23  CONTINUE
       IF(J.GT.1)GO TO 30
       DO 24 I=1,N
       IF(.NOT.REJT(I))IB=I
   24  CONTINUE
       GO TO 35
   30  CONTINUE
       X=URAND(INIT)
       L=INT(1000.*X)
       J=MOD(L,J)
       K=0
       DO 31 I=1,N
       IF(REJT(I))GO TO 31
       IF(K.EQ.J)IB=I
       K=K+1
   31  CONTINUE
   35  CONTINUE
       DO 36 I=1,N
   36  REJT(I)=ELIM(I)
       X=URAND(INIT)
       GO TO 6
C
C   END OF TOURNAMENT
C
   40  CONTINUE
       DO 41 I=1,N
   41  REJT(I)=ELIM(I)
       DO 42 I=1,N
       DO 42 J=1,N
       IF((SCORE(I).LT.SCORE(J)).AND.(.NOT.REJT(J)))
     1 REJT(I)=.TRUE.
   42  CONTINUE
       DO 43 I=1,N
       DO 43 J=1,N
       IF((I.EQ.J).OR.(REJT(I)).OR.(REJT(J)))GO TO 43
       IF(LASTW(I).LT.LASTW(J))GO TO 44
       REJT(J)=.TRUE.
       GO TO 43
   44  REJT(I)=.TRUE.
   43  CONTINUE
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      DO 45 I=1,N
      IF(.NOT.REJT(I))  WIN=I
 45   CONTINUE
      WRITE(6,220)(PLAYV(J),J=1,N)
      WRITE(6,219)(SCORE(I),I=1,N)
      WRITE(6,211)WIN,NO
      XS=XS+NLFT/FLOAT(MANY)
      IF(WIN.EQ.1)XW=XW+1./MANY
      XL=XL+(M-NO)/FLOAT(MANY)
100   CONTINUE
      WRITE(6,221)XS,XW,XL
      WRITE(6,210)INIT
      STOP
200   FORMAT(I10)
201   FORMAT(2I4)
202   FORMAT(10F8.5)
203   FORMAT(F10.5)
215   FORMAT(' ',I4,' BEATS ',I4,' IN GAME ',I4)
216   FORMAT(' ',30X,'X= ',F10.8,' INIT= ',I10)
210   FORMAT('0',I10)
211   FORMAT('0WINNER IS ',I4,' AFTER ',I4,' GAMES')
217   FORMAT(' ',60X,I4,' ELIMINATED AFTER GAME ',I4)
218   FORMAT('1 TOURNAMENT WITH ',I8,' PLAYERS FOR
     1 UP TO ',I8,' GAMES '///)
219   FORMAT(' ',80X,'SCORES ',10I4)
220   FORMAT(' ',80X,'PLAYS  ',10I4)
221   FORMAT('1SUBSET',F8.3,' SUCCESS ',F8.3,' SAVING '
     1 ,F8.3)
      END
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